Helix Waveguide

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Helix waveguide, composed of closely wound turns of insulated copper wire covered with a lossy jacket, shows great promise for use as a communication medium. The properties of this type of waveguide have been investigated using the sheath helix model. Modes whose wall currents follow the highly conducting helix have attenuation constants which are essentially the same as for copper pipe. The other modes have very large attenuation constants which depend upon the helix pitch angle and the electrical properties of the jacket. Approximate formulas are given for the propagation constants of the lossy modes. The circular electric mode important for long-distance communication has low loss for zero-pitch helices. The propagation constants of some of the lossy modes in helix waveguide of zero pitch have been calculated numerically, as functions of the jacket parameters and the guide size, in regions where the approximate formulas are no longer valid. Under certain conditions the attenuation constant of a particular mode may pass through a maximum as the jacket conductivity is varied.

Glossary of symbols

a	Inner radius of waveguide
$h = \beta - i\alpha$	Complex phase constant
n	Angular mode index
p	Denotes p_{nm} or p_{nm}' according to context
p_{nm}	m^{th} zero of $J_n(x)$
p_{nm}'	m^{th} zero of $J_n'(x)$
r, θ, z	Right-handed cylindrical coordinates
α	Attenuation constant
β	Phase constant
$\beta_0 = 2\pi/\lambda_0 = \omega(\mu_0\epsilon_0)^{1/2}$	Free-space phase constant
ϵ_0	Permittivity of interior medium
ϵ	Permittivity of exterior medium
ϵ'	ϵ/ϵ_0
ϵ''	$\sigma/\omega\epsilon_0$

ζ_1	$[\omega^2\mu_0\epsilon_0 - h^2]^{1/2}$
ζ_2	$[\omega^2\mu_0\epsilon_0(\epsilon' - i\epsilon'') - h^2]^{1/2}$
λ_0	Free-space wavelength
$\lambda_c = 2\pi a/p$	Cutoff wavelength
μ_0	Permeability of interior and exterior media
$\nu = \lambda_0/\lambda_c = p\lambda_0/2\pi a$	Cutoff ratio
$\xi + i\eta$	$\dfrac{(\epsilon' - 1 + \nu^2 - i\epsilon'')^{1/2}}{\epsilon' - i\epsilon''}$
Π	Electric Hertz vector
Π^*	Magnetic Hertz vector
σ	Conductivity of exterior medium
ψ	Pitch angle of helix
ω	Angular frequency
$e^{i\omega t}$	Harmonic time dependence assumed throughout
$J_n(x)$	Bessel function of the first kind
$J_n'(x)$	$dJ_n(x)/dx$
$H_n^{(2)}(x)$	Hankel function of the second kind
$H_n^{(2)'}(x)$	$dH_n^{(2)}(x)/dx$

MKS rationalized units are employed throughout. Superscripts i and e are used to indicate the interior and exterior regions.

I. INTRODUCTION AND SUMMARY

Propagation of the lowest circular electric mode (TE_{01}) in cylindrical pipe waveguide holds great promise for low-loss long distance communication.[1, 2] For example, the TE_{01} mode has a theoretical heat loss of 2 db/mile in waveguide of diameter 6 inches at a frequency of 5.5 kmc/s, and the loss decreases with increasing frequency. Increased transmission bandwidth, reduced delay distortion, and reduced waveguide size for a given attenuation are factors favoring use of the highest practical frequency of operation. An increased number of freely propagating modes and smaller mechanical tolerances are the associated penalties. Any deviation of the waveguide from a straight circular cylinder gives rise to signal distortions because of mode conversion-reconversion effects.

One solution to mode conversion-reconversion problems is to obtain a waveguide having the desired low attenuation properties of the TE_{01} mode in metallic cylindrical waveguide and very large attenuation for all other modes, the unwanted modes.[1, 3] The low loss of the circular electric modes in ordinary round guide is the result of having only cir-

[1] S. E. Miller, B.S.T.J., **33**, pp. 1209–1265, 1954.
[2] S. E. Miller and A. C. Beck, Proc. I.R.E., **41**, pp. 348–358, 1953.
[3] S. E. Miller, Proc. I.R.E., **40**, pp. 1104–1113, 1952.

cumferential current flow at the houndary wall. All other modes in round guide have a longitudinal current present at the wall. Thus the desired attenuation properties can he ohtained hy providing a highly conducting circumferential path and a resistive longitudinal path for the wall currents. This is done in the spaced-disk line hy sandwiching lossy layers hetween coaxially arranged annular copper disks.[4] Another possihility which has heen suggested is a helix having a small pitch.

Helix waveguide, formed hy winding insulated wire on a removahle mandrel and coating the helix with lossy material, has heen made at the Holmdel Radio Research Lahoratory. Wires of various cross sections and sizes have heen used to wind helices varying from $\frac{7}{16}$ to 5 inches in diameter, which have heen tested at frequencies from 9 to 60 kmc/s. Pitch angles of from nearly 0° (wire in a plane perpendicular to the axis of propagation) to 90° (wire parallel to the axis of propagation) have heen used. The helices having the highest attenuation for the unwanted modes while maintaining low loss for the TE_{01} mode are those wound with the smallest pitch from insulated wire of diameter 10 to 3 mils (American Wire Gauge Nos. 30 to 40). The high attenuation properties for unwanted modes also depend markedly on the electrical properties of the jacket surrounding the helix.

In this paper the normal modes of helix waveguide are determined using the sheath helix approximation, a mathematical model in which the helical winding is replaced hy an anisotropic conducting sheath. A brief formulation of the boundary value prohlem leads to an equation which determines the propagation constants of modes in the helix guide. Since the equation is not easy to solve numerically, approximations are presented which show the effects of the pitch angle, the diameter, the conductivity and dielectric constant of the jacket, and the wavelength, when the eonductivity of the jacket is sufficiently high.

By proper choice of the pitch angle and, in some instances, of the polarization, a helix waveguide can he made to propagate any mode of ordinary round guide, with an attenuation constant which should he essentially the same as in solid copper pipe. The pitch is chosen so that the wall currents associated with the desired mode follow the direction of the conducting wires. The losses to the other modes are in general much higher, and are determined hy hoth the pitch angle and the jacket material.

Special attention is given in the present work to the limiting case of a helix of zero pitch, since the attenuation constant of the TE_{01} mode will he smallest when the pitch angle is as small as possihle. To explore the

[4] Reference 3, p. 1111.

region where the approximate formulas for the propagation constants of the lossy modes hreak down, some numerical results have heen obtained for helices of zero pitch using an IBM 650 magnetic drum calculator. Tables and curves are given showing the propagation constants of various modes in such a waveguide, as functions of the electrical properties of the jacket and for three different ratios of radius/wavelength. In many cases it is fouud that the attenuation constant of a given mode passes through a maximum as the jacket conductivity is varied, the other parameters remaining fixed. The numerical calculations indicate that it is possihle to get unwanted mode attenuations several hundred to several hundred thousand times greater than the TE_{01} attenuation for the size waveguide that looks most promising for low-loss communication.

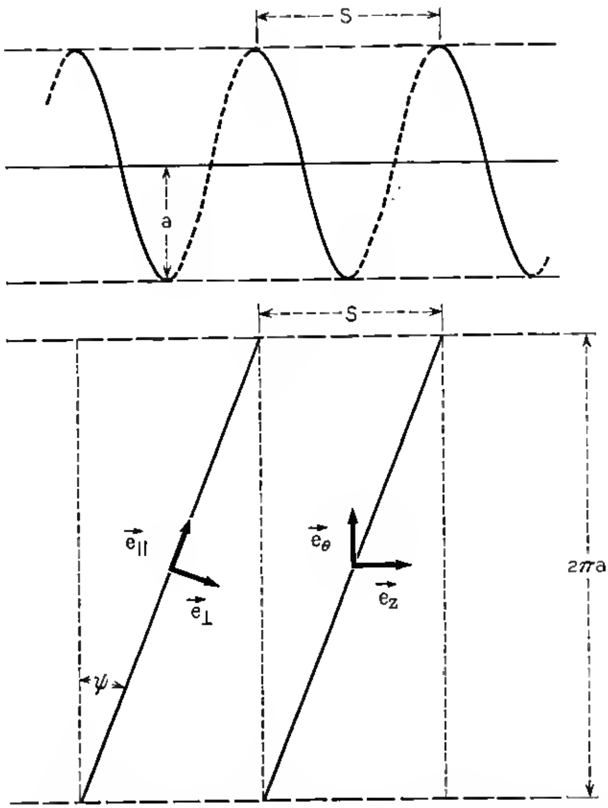

Fig. 1 — Schematic diagrams of the helical sheath and the helical sheath developed, showing the unit vectors and the periodicity.

II. SHEATH HELIX BOUNDARY VALUE PROBLEM

Ordinary cylindrical waveguide consists of a circular cylinder of radius a, infinite length, and zero (or very small) conductivity, imhedded in an infinite* homogeneous conducting medium. The sheath helix waveguide has the same configuration plus the additional property that at radius a dividing the two media, there is an anisotropic conducting sheath which conducts perfectly in the helical direction and does not conduct in the perpendicular direction. The attenuation and phase constants are determined by solving Maxwell's equations in cylindrical coordinates and matching the electric and magnetic fields at the wall of the guide.

The helix of radius a and pitch angle $\psi = \tan^{-1} s/2\pi a$ is shown in the upper part of Fig. 1. The developed helix as viewed from the inside when cut by a plane of constant θ and unrolled is shown in the lower part of the illustration. A new set of unit vectors $\vec{e}_\parallel$ and $\vec{e}_\perp$ parallel and perpendicular respectively to the helix direction is introduced. These are related to $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_z$ by

$$\vec{e}_r \times \vec{e}_\parallel = \vec{e}_\perp$$

$$\vec{e}_\parallel = \vec{e}_z \sin\psi + \vec{e}_\theta \cos\psi$$

$$\vec{e}_\perp = \vec{e}_z \cos\psi - \vec{e}_\theta \sin\psi$$

The boundary conditions at $r = a$ are

$$E_\parallel^{\,i} = E_\parallel^{\,e} = 0$$

$$E_\perp^{\,i} = E_\perp^{\,e}$$

$$H_\parallel^{\,i} = H_\parallel^{\,e}$$

where the superscript i refers to the interior region, $0 \leqq r \leqq a$, and the superscript e refers to the exterior region, $a \leqq r \leqq \infty$. An equivalent set of boundary conditions in terms of the original unit vectors is

$$\begin{aligned} E_z^{\,i} \tan\psi + E_\theta^{\,i} &= 0 \\ E_z^{\,e} \tan\psi + E_\theta^{\,e} &= 0 \\ E_z^{\,i} &= E_z^{\,e} \\ H_z^{\,i} \tan\psi + H_\theta^{\,i} &= H_z^{\,e} \tan\psi + H_\theta^{\,e} \end{aligned} \tag{1}$$

We are looking for solutions which are similar to the modes of or-

* The assumption of an infinite external medium is made to simplify the mathematics. The results will be the same as for a finite conducting jacket which is thick enough so that the fields at its outer surface are negligible.

dinary waveguide, i.e., "fast" modes as contrasted with the well-known "slow" modes used in traveling-wave tubes.[5, 6] To solve the problem we follow the procedure set up by Stratton[7] for the ordinary cylindrical waveguide boundary problem. The fields $\vec{E}$ and $\vec{H}$ are derived from an electric Hertz vector $\vec{\Pi}$ and a magnetic Hertz vector $\vec{\Pi}^*$ by

$$\begin{aligned} \vec{E} &= \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi} - i\omega\mu\vec{\nabla} \times \vec{\Pi}^* \\ \vec{H} &= (\sigma + i\omega\epsilon)\vec{\nabla} \times \vec{\Pi} + \vec{\nabla} \times \vec{\nabla} \times \vec{\Pi}^* \end{aligned} \tag{2}$$

where

$$\begin{aligned} \vec{\Pi} &= \vec{e}_z\Pi_z \\ \vec{\Pi}^* &= \vec{e}_z\Pi^* \end{aligned} \tag{3}$$

and, assuming a time dependence exp $(i\omega t)$,

$$\begin{aligned} \Pi_z^{\,i} &= \sum_{n=-\infty}^{\infty} a_n^{\,i} J_n(\zeta_1 r) e^{-ihz-in\theta} \\ \Pi_z^{\,e} &= \sum_{n=-\infty}^{\infty} a_n^{\,e} H_n^{(2)}(\zeta_2 r) e^{-ihz-in\theta} \\ \Pi_z^{*i} &= \sum_{n=-\infty}^{\infty} b_n^{\,i} J_n(\zeta_1 r) e^{-ihz-in\theta} \\ \Pi_z^{*e} &= \sum_{n=-\infty}^{\infty} b_n^{\,e} H_n^{(2)}(\zeta_2 r) e^{-ihz-in\theta} \end{aligned} \tag{4}$$

In these expressions

$$\begin{aligned} \zeta_1^2 &= \omega^2\mu_0\epsilon_0 - h^2 \\ \zeta_2^2 &= \omega^2\mu_0\epsilon_0(\epsilon' - i\epsilon'') - h^2 \\ \epsilon' - i\epsilon'' &= \epsilon/\epsilon_0 - i\sigma/\omega\epsilon_0 \end{aligned}$$

where the interior region is assumed to have permittivity ϵ_0 and permeability μ_0, while the exterior region has permittivity ϵ, permeability μ_0, and conductivity σ. The superscripts i and e refer to the interior and exterior regions respectively, and the a's and b's are amplitude coefficients.

[5] J. R. Pierce, Proc. I.R.E., **35**, pp. 111–123, 1947.
[6] S. Sensiper, Electromagnetic Wave Propagation on Helical Conductors, Sc.D. thesis, M.I.T., 1951. In Appendix B of this reference, Sensiper shows that when the interior and exterior media are the same, only slow waves will exist except in special cases. Fast guided waves become possible if the conductivity of the exterior medium is sufficiently high.
[7] J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, pp. 524–527. Note that Stratton uses the time dependence exp $(-i\omega t)$.

Attention is restricted to waves traveling in the positive z-direction, which are represented by the factor $\exp(-ihz)$, where h $(=\beta - i\alpha)$ is the complex phase constant. However it is necessary to consider both right and left circularly polarized waves; this accounts for the use of both positive and negative values of n.

Substitution of (2), (3), and (4) into the boundary conditions (1) leads to the following set of equations:

$$\left[\zeta_1^2 \tan\psi - \frac{hn}{a}\right] J_n(\zeta_1 a)a_n^i + i\omega\mu_0\zeta_1 J_n'(\zeta_1 a)b_n^i = 0$$

$$\left[\zeta_2^2 \tan\psi - \frac{hn}{a}\right] H_n^{(2)}(\zeta_2 a)a_n^e + i\omega\mu_0\zeta_2 H_n^{(2)\prime}(\zeta_2 a)b_n^e = 0$$

$$\zeta_1^2 J_n(\zeta_1 a)a_n^i - \zeta_2^2 H_n^{(2)}(\zeta_2 a)a_n^e = 0 \tag{5}$$

$$-i\omega\epsilon_0\zeta_1 J_n'(\zeta_1 a)a_n^i + \left[\zeta_1^2 \tan\psi - \frac{hn}{a}\right] J_n(\zeta_1 a)b_n^i$$

$$+ (\sigma + i\omega\epsilon)\zeta_2 H_n^{(2)\prime}(\zeta_2 a)a_n^e - \left[\zeta_2^2 \tan\psi - \frac{hn}{a}\right] H_n^{(2)}(\zeta_2 a)b_n^e = 0$$

If the conductivity of the exterior region is infinite, it is possible to satisfy the boundary conditions with only one of the amplitude coefficients different from zero; for example

$$\begin{array}{rcr} b_n^i = a_n^e = b_n^e = 0 & & a_n^i = a_n^e = b_n^e = 0 \\ a_n^i \neq 0 & \text{or} & b_n^i \neq 0 \\ J_n(\zeta_1 a) = 0 & & J_n'(\zeta_1 a) = 0 \end{array}$$

The first case corresponds to TM modes and the second to TE modes in a perfectly conducting circular guide. Linearly polarized modes may be represented as combinations of terms in a_n^i and a_{-n}^i, or b_n^i and b_{-n}^i.

If the exterior region is not perfectly conducting, one can still find solutions having the fields confined to the interior region by properly choosing the angle of the perfectly conducting helical sheath. For example, it is easy to verify that equations (5) are satisfied under the following conditions:

$$a_n^i = a_n^e = b_n^e = 0$$

$$b_n^i \neq 0$$

$$\tan\psi = \frac{hn}{\zeta_1^2 a}$$

$$J_n'(\zeta_1 a) = 0$$

If $n \neq 0$, these conditions correspond to circularly polarized TE_{nm} waves, in which the wall currents follow the direction of the conducting sheath. If $n = 0$, then $\psi = 0$, and one has TE_{0m} modes with circumferential currents only.

The equations can also be satisfied with

$$\begin{aligned} b_n^{\,i} = a_n^{\,e} = b_n^{\,e} &= 0 \\ a_n^{\,i} &\neq 0 \\ \psi &= 90° \\ J_n(\zeta_1 a) &= 0 \end{aligned}$$

corresponding to the TM_{nm} modes (either circularly or linearly polarized) of a perfectly conducting pipe, which are associated with longitudinal wall currents only.

In the general case when the jacket is not perfectly conducting and the helix pitch angle is not restricted to special values, it is necessary to solve (5) simultaneously for the field amplitudes. The equations admit a nontrivial solution if and only if the determinant of the coefficients of the a's and b's vanishes. The transcendental equation which results from equating the determinant of the coefficients to zero is

$$\begin{aligned} &\zeta_2 \left[\left(\zeta_1 \tan \psi - \frac{hn}{\zeta_1 a} \right)^2 \frac{J_n(\zeta_1 a)}{J_n'(\zeta_1 a)} - \omega^2 \mu_0 \epsilon_0 \frac{J_n'(\zeta_1 a)}{J_n(\zeta_1 a)} \right] \\ &= \zeta_1 \left[\left(\zeta_2 \tan \psi - \frac{hn}{\zeta_2 a} \right)^2 \frac{H_n^{(2)}(\zeta_2 a)}{H_n^{(2)\prime}(\zeta_2 a)} - \omega^2 \mu_0 \epsilon_0 (\epsilon' - i\epsilon'') \frac{H_n^{(2)\prime}(\zeta_2 a)}{H_n^{(2)}(\zeta_2 a)} \right] \end{aligned} \tag{6}$$

The solution of this equation determines the propagation constant ih and therefore the attenuation and phase constants α and β. When ih has been obtained, it is a straightforward matter to determine the a and b coefficients from equations (5) and the electric and magnetic fields from (2), (3), and (4).

It is well known[8] that the only pure TE or TM modes that can exist in a circular waveguide with walls of finite conductivity are the circularly symmetric TE_{0m} and TM_{0m} modes. The other modes are all mixed modes whose fields are not transverse with respect to either the electric or the magnetic vector. In general the modes of helix waveguide are also mixed modes, and no entirely satisfactory scheme for labeling them has been proposed. In the present paper we shall call the modes TE_{nm} or TM_{nm} according to the limits which they approach as the jacket conductivity becomes infinite, even though they are no longer transverse and their

[8] Reference 7, p. 526.

field patterns may be quite different when the jacket is lossy. This system is not completely unambiguous, because as will appear in Section IV the mode designations thus obtained are not always unique. However it is a satisfactory way to identify the modes so long as the jacket conductivity is high enough for the loss to be treated as a perturbation. Approximations derived on this basis are presented in the next section.

III. APPROXIMATE EXPRESSIONS FOR PROPAGATION CONSTANTS

If the jacket were perfectly conducting, the helix waveguide modes would be the same as in an ideal circular waveguide, with propagation constants given by

$$ih = i\beta_{nm} = i(2\pi/\lambda_0)(1 - \nu^2)^{1/2}$$

where

$\nu = \lambda_0/\lambda_c = p\lambda_0/2\pi a$

$p = m^{\text{th}}$ zero of $J_n(x)$ for TM_{nm} mode, or m^{th} zero of $J_n'(x)$ for TE_{nm} mode

If the jacket conductivity is sufficiently large, approximate solutions of (6) may be found by replacing $H_n^{(2)}(\zeta_2 a)$ and $H_n^{(2)\prime}(\zeta_2 a)$ with their asymptotic expressions, and expanding $J_n(\zeta_1 a)$ or $J_n'(\zeta_1 a)$ in a Taylor series near a particular zero. This calculation is carried out in the appendix. The propagation constant may be written in the form

$$ih = \alpha + i(\beta_{nm} + \Delta\beta)$$

where to first order the perturbation terms are

TM_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{\xi + i\eta}{a(1 - \nu^2)^{1/2}} \frac{1}{1 + \tan^2 \psi} \tag{7a}$$

TE_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{\xi + i\eta}{a(1 - \nu^2)^{1/2}} \frac{\nu^2 p^2}{p^2 - n^2} \frac{[\tan \psi - n(1 - \nu^2)^{1/2}/p\nu]^2}{1 + \tan^2 \psi} \tag{7b}$$

and

$$\xi + i\eta = (\epsilon' - i\epsilon'')^{-1/2}$$

$$\epsilon' = \epsilon/\epsilon_0, \qquad \epsilon'' = \sigma/\omega\epsilon_0$$

The approximations made in deriving (7) are discussed in the appen-

dix. In practice, the range of validity of these expressions is usually limited by the criterion

$$\frac{a(1 - \nu^2)^{1/2}}{\nu} \left| \alpha + i\Delta\beta \right| \ll 1 \tag{8}$$

The numerical calculations described in Section IV indicate that the approximations are good so long as the left-hand side of (8) is less than about 0.1, and that they break down a little sooner for TE modes than for TM modes.

Inspection of (7) reveals three cases of particular interest, namely $\psi = 0°$, $\psi = \tan^{-1} n(1 - \nu^2)^{1/2}/p\nu$, and $\psi = 90°$. These cases, which were mentioned in Section II and are discussed again below, correspond to preferential propagation of certain modes, in which the wall currents follow the direction of the conducting helix. The preferred modes have zero attenuation in the present treatment because the helical sheath is assumed to be perfectly conducting. In practical helices wound from insulated copper wire the loss should be only slightly greater than in round copper pipe of the same diameter. The slight increase (of magnitude 10 per cent to 30 per cent) is due to the slightly nonuniform current distribution in the wires, an effect that can be kept small by keeping the gaps between the wires of the helix small. In general the attenuation constants of modes whose wall currents do not follow the helix are orders of magnitude larger than the attenuation constants of the preferred modes.

$\psi = 0°$

The circular electric (TE_{0m}) modes have attenuation constants substantially the same as in solid copper pipe. The additional TE_{0m} loss if the pitch angle is not quite zero is proportional to $\tan^2 \psi$. This added loss can be made very small by using fine wire for winding the helix.

The losses for the unwanted modes can be made large by a proper choice of jacket material. When $\psi = 0$, equations (7) yield

TM_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{\xi + i\eta}{a(1 - \nu^2)^{1/2}} \tag{9a}$$

TE_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{(1 - \nu^2)^{1/2}}{a} \frac{n^2}{p^2 - n^2} (\xi + i\eta) \tag{9b}$$

It may be of interest to compare the attenuation constants given by (9) with the results obtained by calculating the power dissipated in the walls of a pipe[9] which has different resistances in the circumferential and longitudinal directions. If the wall resistance for circumferential currents is represented by R_θ and for longitudinal currents by R_z, the expressions for α are

TM_{nm} *modes*

$$\alpha = \frac{R_z}{(\mu_0/\epsilon_0)^{1/2} a(1 - \nu^2)^{1/2}}$$

TE_{nm} *modes*

$$\alpha = \frac{R_\theta \nu^2 + R_z(n/p)^2(1 - \nu^2)}{(\mu_0/\epsilon_0)^{1/2} a(1 - \nu^2)^{1/2}} \frac{p^2}{p^2 - n^2}$$

The results for ordinary metallic pipe are obtained by setting

$$R_\theta = R_z = R = (\omega\mu_0/2\sigma)^{1/2}$$

If $R_\theta = 0$, the expressions above agree with (9), inasmuch as $\xi = R(\epsilon_0/\mu_0)^{1/2}$ when the jacket conductivity is large.

$$\psi = \tan^{-1} n(1 - \nu^2)^{1/2}/p\nu, \; n \neq 0$$

For this value of ψ the circularly polarized TE_{nm} mode which varies as $\exp(-in\theta)$ has low attenuation. (We assume $n \neq 0$, since the case $n = 0$ has been treated above.) One of the properties of helix waveguide is the difference in propagation between right and left circularly polarized TE_{nm} modes. By properly designing the helix angle for the frequency, mode, and size of guide, the loss to one of the polarizations can be made very low. If the jacket is lossy enough the attenuation of the other polarization should be quite high. Thus only one of the circularly polarized modes should be propagated through a long pipe. Such a helix has features analogous to the optical properties of levulose and dextrose solutions, which distinguish between left and right circularly polarized light.

Let α_n be the attenuation constant of the mode which varies as $\exp(-in\theta)$, and α_{-n} the attenuation constant of the mode which varies

[9] S. A. Schelkunoff, Electromagnetic Waves, van Nostrand, New York, 1943, pp. 385–387.

as exp $(+in\theta)$. Then from (7b), for any pitch angle ψ,

$$\alpha_{-n} = \frac{\xi}{a}\frac{p^2}{p^2 - n^2}\frac{\nu^2}{(1-\nu^2)^{1/2}}\frac{[\tan\psi + n(1-\nu^2)^{1/2}/p\nu]^2}{1+\tan^2\psi}$$

$$\alpha_{n} = \frac{\xi}{a}\frac{p^2}{p^2 - n^2}\frac{\nu^2}{(1-\nu^2)^{1/2}}\frac{[\tan\psi - n(1-\nu^2)^{1/2}/p\nu]^2}{1+\tan^2\psi}$$

$$\alpha_{-n} - \alpha_n = 4\frac{\xi}{a}\frac{np}{p^2-n^2}\frac{\nu\tan\psi}{1+\tan^2\psi}$$

The mode which varies as $\exp(-in\theta)$ has lower loss if ψ and n have the same sign.

The TM_{nm} attenuation constants are independent of polarization and are given by (7a).

$\psi = 90°$

These "helices," with wires parallel to the axis of the waveguide, should propagate TM_{nm} modes with losses approximately the same as in copper pipe. For the TE_{nm} modes, (7b) gives

TE_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{\nu^2}{a(1-\nu^2)^{1/2}}\frac{p^2}{p^2-n^2}(\xi + i\eta)$$

IV. NUMERICAL SOLUTIONS FOR ZERO-PITCH HELICES

The main interest in helix waveguide is for small pitch angles where the TE_{01} attenuation is very low. The propagation constants of various lossy modes in helix guides of zero pitch have been calculated by solving the characteristic equation (6) numerically. These calculations will now be described.

Equation (6) is first simplified by setting $\psi = 0$ and replacing the Hankel functions with their asymptotic expressions. The condition for validity of the asymptotic expressions, namely

$$|\zeta_2 a| \gg |(4n^2 - 1)/8|$$

is well satisfied in all cases to be treated here. Equation (6) may then be rearranged in the dimensionless form

$$\begin{aligned} F_n(\zeta_1 a) &= (\zeta_2 a)^3\,[(nha)^2 J_n^2(\zeta_1 a) - (\beta_0 a)^2(\zeta_1 a)^2 J_n'^2(\zeta_1 a)] \\ &\quad - i(\zeta_1 a)^3\,[(nha)^2 + (\beta_0 a)^2(\epsilon' - i\epsilon'')(\zeta_2 a)^2] J_n'(\zeta_1 a) J_n(\zeta_1 a) \qquad (10) \\ &= 0 \end{aligned}$$

There is no difference between the propagation constants of right and

left circularly polarized waves when $\psi = 0$. Using the relationships

$$\zeta_2 a = [(\zeta_1 a)^2 + (\beta_0 a)^2 (\epsilon' - i\epsilon'' - 1)]^{1/2}, \qquad \mathrm{Im}\,\zeta_2 a < 0$$
$$ha = [(\beta_0 a)^2 - (\zeta_1 a)^2]^{1/2}, \qquad \mathrm{Im}\, ha < 0$$

it is clear that $F_n(\zeta_1 a)$ is an even function of $\zeta_1 a$, involving the parameters $\beta_0 a$ $(= 2\pi a/\lambda_0)$, ϵ', ϵ'', and n.

When specific values have been assigned to $\beta_0 a$, ϵ', and ϵ'', roots of (10) can be found numerically by the straightforward procedure of evaluating $F_n(\zeta_1 a)$ at a regular network of points in the plane of the complex variable $\zeta_1 a$, plotting the families of curves $\mathrm{Re}\, F_n = 0$ and $\mathrm{Im}\, F_n = 0$, and reading off the values of $\zeta_1 a$ corresponding to the intersections of curves of the two families.

The procedure just outlined has been applied to the cases $n = 0$ and $n = 1$. When $n = 0$ one can take out of $F_0(\zeta_1 a)$ the factor $J_0'(\zeta_1 a)$, whose roots correspond to the TE_{0m} modes; the roots of the other factor are the TM_{0m}-limit modes. When $n = 1$ the function $F_1(\zeta_1 a)$ does not factor, and its roots correspond to both TE_{1m}-limit and TM_{1m}-limit modes. If the jacket conductivity is high it is easy to identify the various limit modes, and a given mode can be traced continuously if the conductivity is decreased in sufficently small steps.

The numerical calculations were set up, more or less arbitrarily, to cover the region $0 \leqq \mathrm{Re}\, \zeta_1 a \leqq 10$, $-10 \leqq \mathrm{Im}\, \zeta_1 a \leqq 10$, for each set of parameter values. A few plots of $\mathrm{Re}\, F_n$ and $\mathrm{Im}\, F_n$ made it apparent that for propagating modes the roots in this region are all in the first quadrant and usually near the real axis. The entire process of solution was then programmed by Mrs. F. M. Laurent for automatic execution on an IBM 650 magnetic drum calculator. The calculator first evaluated $F_n(\zeta_1 a)$ at a network of points spaced half a unit apart in both directions, then examined the sign changes of $\mathrm{Re}\, F_n$ and $\mathrm{Im}\, F_n$ around each elementary square. If it appeared that a particular square might contain a root of F_n, the values of F_n at the four corner points were fitted by an interpolating cubic polynomial[10] which was then solved. If the cubic had a root inside the given square, this was recorded as an approximate root of F_n. The normalized propagation constant $iha = \alpha a + i\beta a$ was also recorded for each root.

The calculated roots $\zeta_1 a$ and the normalized propagation constants are summarized in Tables I(a) to I(f), which relate to the following cases:

Table I(a) — $\beta_0 a = 29.554$, $\epsilon' = 4$, ϵ'' variable

Table I(b) — $\beta_0 a = 29.554$, $\epsilon' = 100$, ϵ'' variable

Table I(c) — $\beta_0 a = 29.554$, $\epsilon' = \epsilon''$, both variable

[10] A. N. Lowan and H. E. Salzer, Jour. Math. and Phys., **23**, p. 157, 1944.

Table I(d) — $\beta_0 a = 12.930$, $\epsilon' = 4$, ϵ'' variable

Table I(e) — $\beta_0 a = 12.930$, $\epsilon' = \epsilon''$, both variable

Table I(f) — $\beta_0 a = 6.465$, $\epsilon' = 4$, ϵ'' variable

The three values of $\beta_0 a$ correspond to waveguides of diameter 2 inches, $\frac{7}{8}$ inch, and $\frac{7}{16}$ inch at $\lambda_0 = 5.4$ mm. The jacket materials (mostly carbon-loaded resins) which have been tested to date show a range of relative permittivities roughly from 4 to 100. There is some indication that the permittivity of a carbon-loaded resin increases as its conductivity increases; this suggested consideration of the case $\epsilon' = \epsilon''$.

The tables cover the range from $\epsilon'' = 1000$ down to $\epsilon'' = 1$ at small enough intervals so that the general course of each mode can be followed. It is worth noting that at 5.4 mm a resistivity $(1/\sigma)$ of 1 ohm cm corresponds to $\epsilon'' = 32$. Copper at this frequency has an ϵ'' of approximately 2×10^7.

In general the tables include the modes derived from $F_0(\zeta_1 a)$ whose limits are TM_{01}, TM_{02}, and TM_{03}, and the modes derived from $F_1(\zeta_1 a)$ whose limits are TE_{11}, TM_{11}, TE_{12}, TM_{12}, and TE_{13} (except that in the $\frac{7}{16}$-inch guide TM_{03}, TM_{12}, and TE_{13} are cut off). Some results are given for the TM_{13}-limit mode, namely those which satisfy the arbitrary criterion $\text{Re}\,\zeta_1 a \leqq 10$; but these results are incomplete because for large ϵ'' the corresponding root of $F_1(\zeta_1 a)$ approaches 10.173. Furthermore for small values of ϵ'' the attenuation constants of a few of the TM-limit modes become quite large and the corresponding values of $\zeta_1 a$ move far away from the origin. Since our object was to make a general survey rather than to investigate any particular mode exhaustively, we did not attempt to pursue these modes outside the region originally proposed for study.

The results of the IBM calculations are recorded in Table I to three decimal places. Since the roots $\zeta_1 a$ were obtained by cubic interpolation in a square of side 0.5, the last place is not entirely reliable; but spot checks on a few of the roots by successive approximations indicate that it is probably not off by more than one or two units. The propagation constants of some of the relatively low-loss modes (especially TE_{12} and TE_{13}, whose wall currents are largely circumferential) were calculated from the approximate formulas,* as noted in the tables. The attenuation

(*Text continued on page 1375*)

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* The formulas used were (A9) and (A10) of the appendix, which are slightly more accurate than (7) of the text.

TABLE I(a) — 2-INCH GUIDE AT $\lambda_0 = 5.4$ MM ($\beta_0 a = 29.554$) WITH $\epsilon' = 4$ AND ϵ'' VARIABLE

Limit Mode	ϵ''	ha	$\alpha a + i\beta a$
TM_{01}	∞	2.405	29.456i
	1000	2.154 + 0.384i	0.028 + 29.478i
	250	2.094 + 0.974i	0.069 + 29.496i
	100	2.408 + 1.679i	0.137 + 29.504i
	90	2.482 + 1.772i	0.149 + 29.503i
	80	2.579 + 1.878i	0.164 + 29.502i
	64	2.804 + 2.083i	0.198 + 29.495i
	40	3.519 + 2.547i	0.304 + 29.456i
	25	4.604 + 3.165i	0.496 + 29.369i
	16	5.870 + 3.763i	0.756 + 29.219i
	10	7.564 + 4.131i	1.082 + 28.887i
	8	8.464 + 4.158i	1.229 + 28.646i
TM_{02}	∞	5.520	29.034i
	1000	5.399 + 0.127i	0.024 + 29.057i
	250	5.274 + 0.268i	0.049 + 29.081i
	100	5.109 + 0.445i	0.078 + 29.113i
	90	5.081 + 0.472i	0.082 + 29.118i
	80	5.047 + 0.504i	0.087 + 29.125i
	64	4.968 + 0.569i	0.097 + 29.139i
	40	4.716 + 0.701i	0.113 + 29.184i
	25	4.375 + 0.677i	0.101 + 29.237i
	16	4.172 + 0.551i	0.079 + 29.264i
	10	4.047 + 0.448i	0.062 + 29.279i
	8	4.004 + 0.412i	0.056 + 29.285i
	4	3.905 + 0.344i	0.046 + 29.297i
	1	3.820 + 0.310i	0.040 + 29.308i
TM_{03}	∞	8.654	28.259i
	1000	8.577 + 0.078i	0.024 + 28.282i
	250	8.500 + 0.160i	0.048 + 28.306i
	100	8.408 + 0.260i	0.077 + 28.334i
	90	8.395 + 0.275i	0.081 + 28.338i
	80	8.378 + 0.293i	0.086 + 28.343i
	64	8.344 + 0.330i	0.097 + 28.354i
	40	8.253 + 0.424i	0.123 + 28.382i
	25	8.125 + 0.545i	0.156 + 28.421i
	16	7.943 + 0.678i	0.189 + 28.475i
	10	7.658 + 0.779i	0.209 + 28.556i
	8	7.511 + 0.780i	0.205 + 28.595i
	4	7.200 + 0.693i	0.174 + 28.673i
	1	6.986 + 0.612i	0.149 + 28.724i
TE_{11}	∞	1.841	29.497i
	1000	1.703 + 0.234i	0.014 + 29.506i
	250	1.764 + 0.630i	0.038 + 29.508i
	100	2.465 + 0.963i	0.081 + 29.467i
	90	2.660 + 0.748i	0.068 + 29.444i
	80	2.633 + 0.604i	0.054 + 29.443i
	64	2.594 + 0.464i	0.041 + 29.444i
	40	2.546 + 0.312i	0.027 + 29.446i
	25	2.508 + 0.226i	0.019 + 29.448i
	16	2.481 + 0.176i	0.015 + 29.450i
	10	2.455 + 0.140i	0.012 + 29.452i
	8	2.445 + 0.129i	0.011 + 29.453i
	4	2.418 + 0.106i	0.009 + 29.455i
	1	2.394 + 0.095i	0.008 + 29.457i

Limit Mode	ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{11}	∞	3.832	29.305i
	1000	3.652 + 0.197i	0.024 + 29.328i
	250	3.457 + 0.440i	0.052 + 29.355i
	100	2.978 + 0.880i	0.089 + 29.417i
	90	2.821 + 1.215i	0.116 + 29.445i
	80	2.945 + 1.476i	0.148 + 29.444i
	64	3.146 + 1.868i	0.200 + 29.446i
	40	3.728 + 2.564i	0.325 + 29.432i
	25	4.659 + 3.175i	0.504 + 29.361i
	16	5.921 + 3.727i	0.756 + 29.204i
	10	7.613 + 4.135i	1.090 + 28.875i
	8	8.487 + 4.153i	1.231 + 28.639i
TE_{12}	∞	5.331	29.069i
	1000		0.0008 + 29.070i*
	250		0.0016 + 29.071i*
	100		0.0026 + 29.072i*
	64		0.0033 + 29.072i*
	40		0.0042 + 29.073i*
	25		0.0055 + 29.074i*
	10		0.0092 + 29.075i*
	4	5.297 + 0.072i	0.013 + 29.076i
	1	5.322 + 0.096i	0.018 + 29.071i
TM_{12}	∞	7.016	28.710i
	1000	6.918 + 0.099i	0.024 + 28.733i
	250	6.821 + 0.203i	0.048 + 28.757i
	100	6.701 + 0.330i	0.077 + 28.786i
	90	6.683 + 0.349i	0.081 + 28.791i
	80	6.660 + 0.372i	0.086 + 28.796i
	64	6.612 + 0.419i	0.096 + 28.808i
	40	6.475 + 0.535i	0.120 + 28.841i
	25	6.253 + 0.655i	0.142 + 28.893i
	16	5.965 + 0.682i	0.141 + 28.954i
	10	5.719 + 0.590i	0.116 + 29.002i
	8	5.641 + 0.541i	0.105 + 29.016i
	4	5.471 + 0.419i	0.079 + 29.047i
	1	5.317 + 0.347i	0.063 + 29.074i
TE_{13}	∞	8.536	28.295i
	1000		0.0003 + 28.295i*
	250		0.0006 + 28.295i*
	100		0.0010 + 28.296i*
	64		0.0012 + 28.296i*
	40		0.0016 + 28.296i*
	25		0.0020 + 28.296i*
	10		0.0034 + 28.297i*
	4		0.0050 + 28.296i*
	1		0.0058 + 28.295i*
TM_{13}	∞	10.173	27.748i
	100	9.963 + 0.219i	0.078 + 27.825i
	90	9.952 + 0.231i	0.083 + 27.829i
	80	9.938 + 0.246i	0.088 + 27.834i
	64	9.911 + 0.277i	0.098 + 27.845i
	40	9.840 + 0.356i	0.126 + 27.870i
	25	9.746 + 0.460i	0.161 + 27.905i
	16	9.625 + 0.591i	0.204 + 27.950i
	10	9.433 + 0.757i	0.255 + 28.020i
	8	9.305 + 0.837i	0.278 + 28.065i
	4	8.836 + 0.898i	0.281 + 28.218i
	1	8.485 + 0.781i	0.234 + 28.322i

* Approximate formula.

TABLE I(b) — 2-INCH GUIDE AT $\lambda_0 = 5.4$ MM ($\beta_0 a = 29.554$) WITH $\epsilon' = 100$ AND ϵ'' VARIABLE

Limit Mode	ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{01}	∞	2.405	29.456i
	1000	2.178 + 0.391i	0.029 + 29.476i
	250	2.291 + 0.885i	0.069 + 29.479i
	100	2.677 + 1.062i	0.097 + 29.452i
	80	2.764 + 1.047i	0.098 + 29.443i
	64	2.834 + 1.019i	0.098 + 29.436i
	40	2.928 + 0.950i	0.094 + 29.424i
	25	2.973 + 0.893i	0.090 + 29.418i
	10	3.004 + 0.831i	0.085 + 29.413i
	4	3.013 + 0.806i	0.083 + 29.411i
	1	3.016 + 0.793i	0.081 + 29.411i
TM_{02}	∞	5.520	29.034i
	1000	5.406 + 0.133i	0.025 + 29.056i
	250	5.339 + 0.298i	0.055 + 29.069i
	100	5.372 + 0.473i	0.087 + 29.066i
	80	5.398 + 0.508i	0.094 + 29.062i
	64	5.429 + 0.535i	0.100 + 29.056i
	40	5.492 + 0.566i	0.107 + 29.045i
	25	5.540 + 0.573i	0.109 + 29.036i
	10	5.589 + 0.569i	0.109 + 29.027i
	4	5.608 + 0.563i	0.109 + 29.023i
	1	5.617 + 0.560i	0.108 + 29.021i
TM_{03}	∞	8.654	28.259i
	1000	8.581 + 0.082i	0.025 + 28.281i
	250	8.537 + 0.179i	0.054 + 28.295i
	100	8.548 + 0.279i	0.084 + 28.292i
	80	8.561 + 0.300i	0.091 + 28.289i
	64	8.575 + 0.317i	0.096 + 28.285i
	40	8.606 + 0.339i	0.103 + 28.276i
	25	8.630 + 0.348i	0.106 + 28.268i
	10	8.658 + 0.352i	0.108 + 28.260i
	4	8.669 + 0.352i	0.108 + 28.257i
	1	8.675 + 0.351i	0.108 + 28.255i
TE_{11}	∞	1.841	29.497i
	1000	1.719 + 0.236i	0.014 + 29.505i
	250	1.871 + 0.504i	0.032 + 29.499i
	100	2.132 + 0.484i	0.035 + 29.481i
	80	2.161 + 0.451i	0.033 + 29.479i
	64	2.178 + 0.420i	0.031 + 29.477i
	40	2.191 + 0.372i	0.028 + 29.475i
	25	2.192 + 0.343i	0.026 + 29.475i
	10	2.190 + 0.316i	0.023 + 29.475i
	4	2.188 + 0.306i	0.023 + 29.475i
	1	2.187 + 0.301i	0.022 + 29.475i

TABLE I(b) — *Continued*

Limit Mode	ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{11}	∞	3.832	29.305i
	1000	3.663 + 0.204i	0.026 + 29.327i
	250	3.579 + 0.485i	0.059 + 29.341i
	100	3.715 + 0.788i	0.100 + 29.331i
	80	3.787 + 0.826i	0.107 + 29.322i
	64	3.856 + 0.843i	0.111 + 29.314i
	40	3.969 + 0.836i	0.113 + 29.299i
	25	4.043 + 0.817i	0.113 + 29.288i
	10	4.100 + 0.777i	0.109 + 29.279i
	4	4.119 + 0.759i	0.107 + 29.276i
	1	4.128 + 0.749i	0.106 + 29.274i
TE_{12}	∞	5.331	29.069i
	1000		0.0008 + 29.070i*
	250		0.0018 + 29.071i*
	100		0.0028 + 29.071i*
	64		0.0032 + 29.070i*
	40		0.0034 + 29.070i*
	25		0.0035 + 29.070i*
	10		0.0036 + 29.070i*
	4		0.0036 + 29.070i*
	1		0.0036 + 29.069i*
TM_{12}	∞	7.016	28.710i
	1000	6.923 + 0.103i	0.025 + 28.732i
	250	6.868 + 0.226i	0.054 + 28.746i
	100	6.885 + 0.355i	0.085 + 28.743i
	80	6.902 + 0.381i	0.092 + 28.740i
	64	6.922 + 0.403i	0.097 + 28.735i
	40	6.965 + 0.429i	0.104 + 28.725i
	25	7.000 + 0.440i	0.107 + 28.717i
	10	7.037 + 0.443i	0.109 + 28.708i
	4	7.051 + 0.441i	0.108 + 28.704i
	1	7.058 + 0.440i	0.108 + 28.703i
TE_{13}	∞	8.536	28.295i
	1000		0.0003 + 28.295i*
	250		0.0007 + 28.295i*
	100		0.0010 + 28.295i*
	64		0.0012 + 28.295i*
	40		0.0013 + 28.295i*
	25		0.0013 + 28.295i*
	10		0.0013 + 28.295i*
	4		0.0013 + 28.295i*
	1		0.0013 + 28.295i*

* Approximate formula.

Table I(c) — 2-inch Guide at $\lambda_0 = 5.4$ mm ($\beta_0 a = 29.554$) with $\epsilon' = \epsilon''$

Limit Mode	ϵ' and ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{01}	∞	2.405	29.456i
	1000	2.338 + 0.341i	0.027 + 29.464i
	250	2.418 + 0.707i	0.058 + 29.464i
	100	2.677 + 1.062i	0.097 + 29.452i
	64	2.925 + 1.226i	0.122 + 29.435i
	40	3.309 + 1.324i	0.149 + 29.399i
	32	3.540 + 1.299i	0.156 + 29.371i
	25	3.787 + 1.162i	0.150 + 29.334i
	16	3.946 + 0.800i	0.108 + 29.301i
	12	3.950 + 0.647i	0.087 + 29.296i
	10	3.946 + 0.573i	0.077 + 29.295i
	4	3.905 + 0.344i	0.046 + 29.297i
	2	3.869 + 0.252i	0.033 + 29.301i
	1	3.820 + 0.185i	0.024 + 29.307i
TM_{02}	∞	5.520	29.034i
	1000	5.469 + 0.136i	0.026 + 29.044i
	250	5.423 + 0.282i	0.053 + 29.054i
	100	5.372 + 0.473i	0.087 + 29.066i
	64	5.337 + 0.624i	0.115 + 29.075i
	40	5.294 + 0.874i	0.159 + 29.090i
	32	5.279 + 1.061i	0.193 + 29.099i
	25	5.319 + 1.367i	0.250 + 29.105i
	16	5.852 + 1.969i	0.397 + 29.039i
	12	6.472 + 2.178i	0.487 + 28.923i
	10	7.026 + 2.198i	0.536 + 28.796i
TM_{03}	∞	8.654	28.259i
	1000	8.620 + 0.085i	0.026 + 28.269i
	250	8.587 + 0.173i	0.052 + 28.280i
	100	8.548 + 0.279i	0.084 + 28.292i
	64	8.521 + 0.355i	0.107 + 28.302i
	40	8.483 + 0.461i	0.138 + 28.315i
	32	8.458 + 0.526i	0.157 + 28.323i
	25	8.425 + 0.611i	0.182 + 28.335i
	16	8.330 + 0.824i	0.242 + 28.369i
	12	8.206 + 1.037i	0.300 + 28.413i
	10	8.034 + 1.240i	0.350 + 28.471i
	4	7.200 + 0.693i	0.174 + 28.673i
	2	7.098 + 0.483i	0.120 + 28.694i
	1	6.998 + 0.349i	0.085 + 28.716i
TE_{11}	∞	1.841	29.497i
	1000	1.810 + 0.190i	0.012 + 29.499i
	250	1.911 + 0.384i	0.025 + 29.495i
	100	2.132 + 0.484i	0.035 + 29.481i
	64	2.270 + 0.453i	0.035 + 29.470i
	40	2.365 + 0.366i	0.029 + 29.462i
	32	2.389 + 0.324i	0.026 + 29.459i
	25	2.406 + 0.281i	0.023 + 29.457i
	16	2.420 + 0.219i	0.018 + 29.456i
	12	2.424 + 0.187i	0.015 + 29.455i
	10	2.424 + 0.169i	0.014 + 29.455i
	4	2.418 + 0.106i	0.009 + 29.455i
	2	2.409 + 0.078i	0.006 + 29.456i
	1	2.394 + 0.056i	0.005 + 29.457i

TABLE I(c)—*Continued*

Limit Mode	ϵ' and ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{11}	∞	3.832	29.305i
	1000	3.759 + 0.203i	0.026 + 29.315i
	250	3.714 + 0.439i	0.056 + 29.323i
	100	3.715 + 0.788i	0.100 + 29.331i
	64	3.797 + 1.070i	0.139 + 29.329i
	40	4.080 + 1.400i	0.195 + 29.305i
	32	4.276 + 1.550i	0.226 + 29.285i
	25	4.586 + 1.661i	0.260 + 29.245i
	16	5.359 + 1.579i	0.291 + 29.109i
	12	5.587 + 1.043i	0.201 + 29.041i
	10	5.560 + 0.859i	0.164 + 29.040i
	4	5.471 + 0.419i	0.079 + 29.047i
	2	5.438 + 0.249i	0.047 + 29.051i
	1	5.444 + 0.131i	0.025 + 29.049i
TE_{12}	∞	5.331	29.069i
	1000		0.0009 + 29.070i*
	250		0.0018 + 29.070i*
	100		0.0028 + 29.071i*
	64		0.0035 + 29.071i*
	40		0.0044 + 29.071i*
	25		0.0055 + 29.072i*
	10		0.0087 + 29.073i*
	4	5.297 + 0.072i	0.013 + 29.076i
	2	5.272 + 0.108i	0.020 + 29.080i
	1	5.198 + 0.132i	0.023 + 29.094i
TM_{12}	∞	7.016	28.710i
	1000	6.971 + 0.107i	0.026 + 28.721i
	250	6.931 + 0.217i	0.052 + 28.731i
	100	6.885 + 0.355i	0.085 + 28.743i
	64	6.852 + 0.457i	0.109 + 28.753i
	40	6.801 + 0.610i	0.144 + 28.768i
	32	6.768 + 0.708i	0.167 + 28.778i
	25	6.720 + 0.850i	0.198 + 28.793i
	16	6.562 + 1.359i	0.309 + 28.850i
	12	6.869 + 2.095i	0.499 + 28.825i
	10	7.322 + 2.374i	0.605 + 28.737i
TE_{13}	∞	8.536	28.295i
	1000		0.0003 + 28.295i*
	250		0.0007 + 28.295i*
	100		0.0010 + 28.295i*
	64		0.0013 + 28.295i*
	40		0.0016 + 28.295i*
	25		0.0021 + 28.295i*
	10		0.0032 + 28.296i*
	4		0.0050 + 28.296i*
	1		0.0094 + 28.295i*
TM_{13}	∞	10.173	27.748i
	25	9.981 + 0.497i	0.178 + 27.823i
	16	9.910 + 0.652i	0.232 + 27.852i
	12	9.841 + 0.785i	0.277 + 27.880i
	10	9.776 + 0.893i	0.313 + 27.907i
	4	8.836 + 0.898i	0.281 + 28.218i
	2	8.656 + 0.596i	0.183 + 28.265i
	1	8.523 + 0.409i	0.123 + 28.302i

* Approximate formula.

TABLE I(d)—$\frac{7}{8}$-INCH GUIDE AT $\lambda_0 = 5.4$ MM ($\beta_0 a = 12.930$) WITH $\epsilon' = 4$ AND ϵ'' VARIABLE

Limit Mode	ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{01}	∞	2.405	12.704i
	1000	2.286 + 0.140i	0.025 + 12.727i
	250	2.183 + 0.324i	0.056 + 12.749i
	100	2.113 + 0.595i	0.098 + 12.771i
	64	2.114 + 0.800i	0.132 + 12.782i
	40	2.185 + 1.072i	0.183 + 12.790i
	25	2.377 + 1.369i	0.255 + 12.786i
	10	3.212 + 1.699i	0.431 + 12.647i
	6.4	3.694 + 1.440i	0.426 + 12.482i
	4.0	3.765 + 1.029i	0.312 + 12.416i
	2.5	3.700 + 0.853i	0.254 + 12.421i
	1.0	3.624 + 0.733i	0.214 + 12.435i
TM_{02}	∞	5.520	11.692i
	1000	5.468 + 0.054i	0.025 + 11.717i
	250	5.416 + 0.111i	0.051 + 11.742i
	100	5.356 + 0.183i	0.083 + 11.770i
	64	5.317 + 0.235i	0.106 + 11.789i
	40	5.266 + 0.308i	0.137 + 11.814i
	25	5.206 + 0.410i	0.180 + 11.844i
	10	5.073 + 0.772i	0.328 + 11.923i
	6.4	5.095 + 1.137i	0.485 + 11.948i
	4.0	5.486 + 1.429i	0.664 + 11.814i
	2.5	5.818 + 1.379i	0.689 + 11.650i
	1.0	6.041 + 1.188i	0.624 + 11.511i
TM_{03}	∞	8.654	9.607i
	1000	8.620 + 0.034i	0.030 + 9.637i
	250	8.587 + 0.069i	0.061 + 9.667i
	100	8.550 + 0.111i	0.098 + 9.701i
	64	8.525 + 0.141i	0.124 + 9.723i
	40	8.494 + 0.183i	0.160 + 9.752i
	25	8.459 + 0.239i	0.207 + 9.785i
	10	8.393 + 0.411i	0.350 + 9.851i
	6.4	8.386 + 0.532i	0.452 + 9.866i
	4.0	8.426 + 0.668i	0.571 + 9.847i
	2.5	8.515 + 0.769i	0.669 + 9.784i
	1.0	8.676 + 0.824i	0.741 + 9.651i
TE_{11}	∞	1.841	12.798i
	1000	1.767 + 0.074i	0.010 + 12.809i
	250	1.717 + 0.191i	0.026 + 12.817i
	100	1.706 + 0.368i	0.049 + 12.822i
	64	1.734 + 0.500i	0.068 + 12.823i
	40	1.857 + 0.656i	0.095 + 12.813i
	25	2.126 + 0.773i	0.129 + 12.778i
	10	2.436 + 0.411i	0.079 + 12.706i
	6.4	2.413 + 0.316i	0.060 + 12.707i
	4.0	2.386 + 0.262i	0.049 + 12.711i
	2.5	2.364 + 0.234i	0.043 + 12.714i
	1.0	2.341 + 0.212i	0.039 + 12.718i

TABLE I(d) — *Continued*

Limit Mode	ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{11}	∞	3.832	12.349i
	1000	3.750 + 0.081i	0.025 + 12.375i
	250	3.676 + 0.171i	0.051 + 12.398i
	100	3.588 + 0.290i	0.084 + 12.426i
	64	3.530 + 0.382i	0.108 + 12.445i
	40	3.447 + 0.516i	0.143 + 12.474i
	25	3.329 + 0.757i	0.201 + 12.519i
	10	3.749 + 1.664i	0.499 + 12.496i
	6.4	4.275 + 1.750i	0.606 + 12.343i
	4.0	4.701 + 1.553i	0.600 + 12.160i
	2.5	4.843 + 1.274i	0.511 + 12.067i
	1.0	4.844 + 1.031i	0.415 + 12.040i
TE_{12}	∞	5.331	11.780i
	1000		0.0007 + 11.780i*
	250		0.0015 + 11.781i*
	100		0.0024 + 11.782i*
	64		0.0030 + 11.782i*
	40		0.0039 + 11.783i*
	25		0.0051 + 11.784i*
	10		0.0085 + 11.785i*
	4		0.0125 + 11.784i*
	1		0.0146 + 11.781i*
TM_{12}	∞	7.016	10.861i
	1000	6.972 + 0.043i	0.027 + 10.889i
	250	6.930 + 0.087i	0.055 + 10.917i
	100	6.883 + 0.141i	0.088 + 10.947i
	64	6.853 + 0.179i	0.112 + 10.967i
	40	6.814 + 0.233i	0.144 + 10.992i
	25	6.769 + 0.305i	0.187 + 11.023i
	10	6.679 + 0.541i	0.326 + 11.090i
	6.4	6.670 + 0.718i	0.431 + 11.109i
	4.0	6.755 + 0.935i	0.570 + 11.080i
	2.5	6.942 + 1.061i	0.671 + 10.981i
	1.0	7.193 + 1.054i	0.700 + 10.819i
TE_{13}	∞	8.536	9.712i
	1000		0.0002 + 9.712i*
	250		0.0005 + 9.712i*
	100		0.0008 + 9.712i*
	64		0.0010 + 9.713i*
	40		0.0012 + 9.713i*
	25		0.0016 + 9.713i*
	10		0.0027 + 9.713i*
	4		0.0040 + 9.713i*
	1		0.0048 + 9.712i*
TM_{13}	∞	10.173	7.980i
	10	9.949 + 0.340i	0.409 + 8.276i
	6.4	9.943 + 0.436i	0.523 + 8.293i
	4.0	9.970 + 0.543i	0.655 + 8.277i

* Approximate formula.

TABLE I(e) — $\frac{7}{8}$-INCH GUIDE AT $\lambda_0 = 5.4$ MM ($\beta_0 a = 12.930$) WITH $\epsilon' = \epsilon''$

Limit Mode	ϵ' and ϵ''	$\zeta_1 a$	$\alpha a + i\beta a$
TM_{01}	∞	2.405	12.704i
	1000	2.360 + 0.141i	0.026 + 12.714i
	250	2.339 + 0.295i	0.054 + 12.720i
	100	2.351 + 0.482i	0.089 + 12.724i
	64	2.382 + 0.608i	0.114 + 12.724i
	40	2.450 + 0.766i	0.148 + 12.720i
	25	2.573 + 0.942i	0.191 + 12.708i
	10	3.052 + 1.244i	0.301 + 12.630i
	4	3.765 + 1.029i	0.312 + 12.416i
	2	3.841 + 0.653i	0.203 + 12.366i
	1	3.768 + 0.438i	0.133 + 12.378i
TM_{02}	∞	5.520	11.692i
	1000	5.497 + 0.058i	0.027 + 11.704i
	250	5.475 + 0.118i	0.055 + 11.715i
	100	5.451 + 0.190i	0.088 + 11.727i
	64	5.435 + 0.241i	0.111 + 11.735i
	40	5.416 + 0.310i	0.143 + 11.746i
	25	5.393 + 0.402i	0.184 + 11.760i
	10	5.338 + 0.701i	0.317 + 11.802i
	4	5.486 + 1.429i	0.664 + 11.814i
	2	6.389 + 1.780i	0.996 + 11.425i
	1	6.901 + 1.040i	0.652 + 11.003i
TM_{03}	∞	8.654	9.607i
	1000	8.639 + 0.037i	0.033 + 9.621i
	250	8.624 + 0.074i	0.067 + 9.635i
	100	8.607 + 0.118i	0.105 + 9.650i
	64	8.596 + 0.148i	0.132 + 9.661i
	40	8.581 + 0.189i	0.168 + 9.675i
	25	8.563 + 0.241i	0.213 + 9.694i
	10	8.512 + 0.393i	0.344 + 9.747i
	4	8.426 + 0.668i	0.571 + 9.847i
	2	8.320 + 1.094i	0.910 + 9.999i
	1	8.812 + 1.915i	1.721 + 9.806i
TE_{11}	∞	1.841	12.798i
	1000	1.810 + 0.072i	0.010 + 12.803i
	250	1.807 + 0.161i	0.023 + 12.804i
	100	1.833 + 0.265i	0.038 + 12.802i
	64	1.870 + 0.330i	0.048 + 12.799i
	40	1.939 + 0.401i	0.061 + 12.790i
	25	2.047 + 0.459i	0.074 + 12.776i
	10	2.295 + 0.414i	0.075 + 12.732i
	4	2.386 + 0.262i	0.049 + 12.711i
	2	2.389 + 0.186i	0.035 + 12.709i
	1	2.369 + 0.129i	0.024 + 12.712i

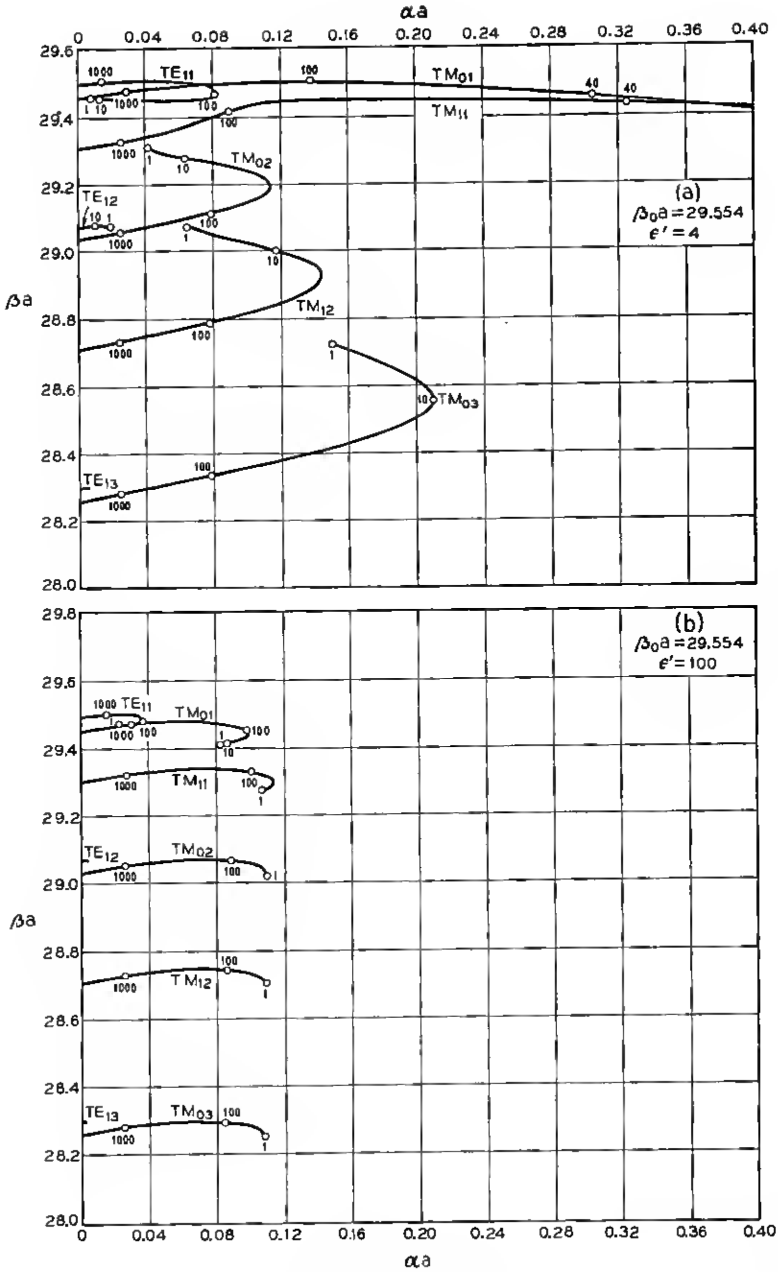


Fig. 2(a) and (b)

Fig. 2 — Plots of phase constant versus attenuation constant for modes in various helix waveguides. Representative values of ϵ'' are shown on the curves.

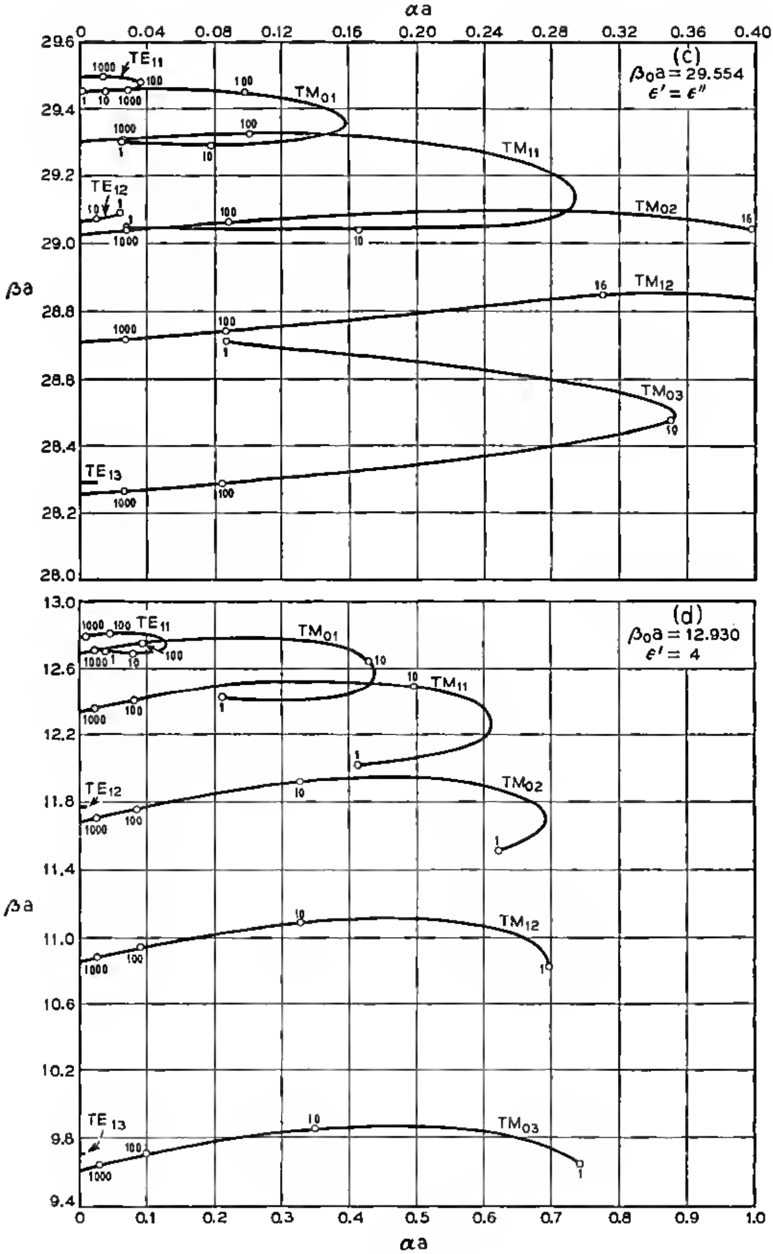
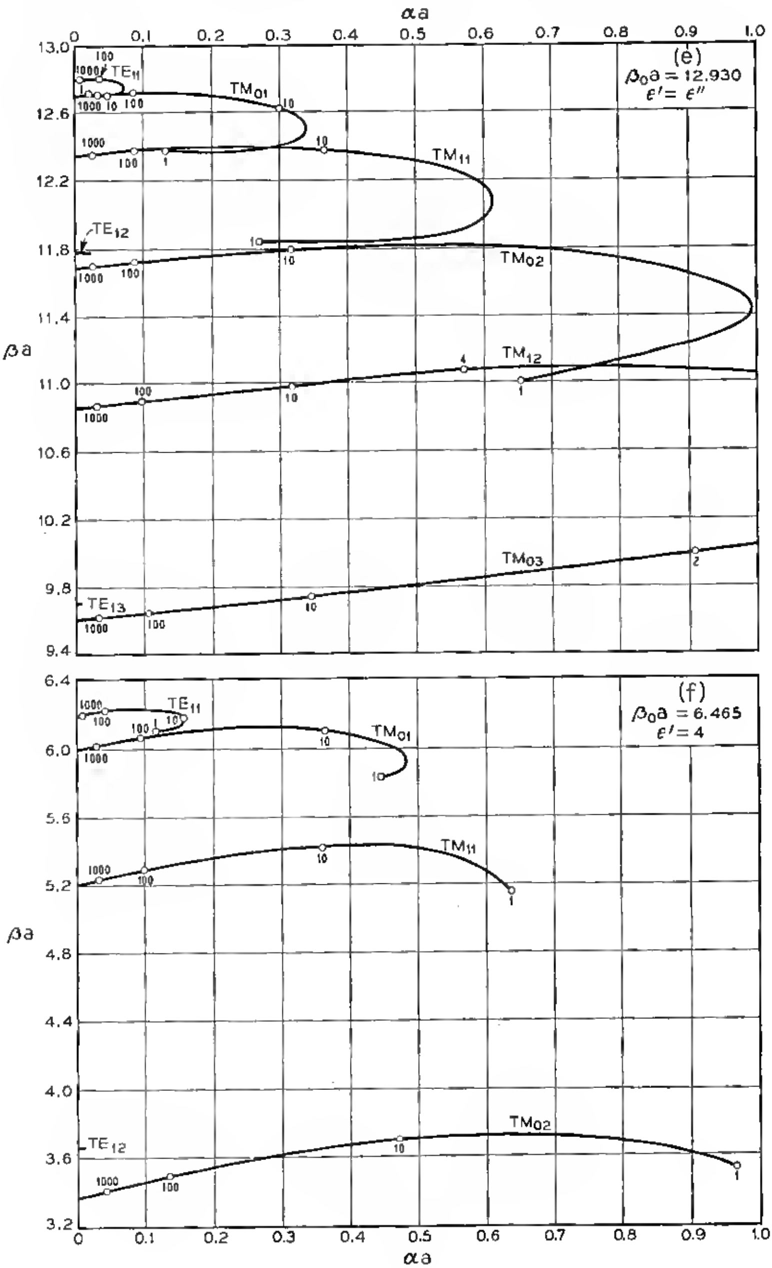

αa
(c)
β₀a = 29.554
ε′ = ε″
TE11
TM01
TM11
TE12
TM02
TM12
TM03
TE13
βa
(d)
β₀a = 12.930
ε′ = 4
αa

Fig. 2(c) and (d)



αa
βa
(e)
β₀a = 12.930
ε′ = ε″
TE11
TM01
TM11
TE12
TM02
TM12
TM03
TE13
(f)
β₀a = 6.465
ε′ = 4
TE11
TM01
TM11
TE12
TM02

Fig. 2(e) and (f)

constants calculated from the approximate formulas are given to four decimal places, i.e., usually two significant figures.

The contents of Table I are displayed graphically in Figs. 2(a) through (f), which show plots of βa vs αa for all modes except TM_{13}. Representative values of ϵ'' are indicated on the curves. Note that the scales are different for the different guide sizes, and that the βa-scale is compressed in all cases. If αa and βa were plotted on the same scale, the curves would make an initial angle of 45° with the αa-axis when $\epsilon' =$ constant, or 22.5° when $\epsilon' = \epsilon''$.

Figs. 3(a) to (f) show the normalized attenuation constants αa of various modes plotted against ϵ'' on a log-log scale. In Fig. 3(b) the curves for all TM modes would be similar to the two shown, and in Fig. 3(d) the TM_{03} curve is like TM_{12}. Although for some modes the attenuation constant increases steadily as the conductivity decreases over the range of our calculations, in many cases the attenuation passes through a maximum and then decreases as the conductivity is further decreased. This phenomenon will be discussed in Section V.

It may be noticed that in some instances the limit modes are not unique. For example, Tables I(a), with $\epsilon' = 4$, and I(c), with $\epsilon' = \epsilon''$, for the large guide have in common the case $\epsilon' = 4$, $\epsilon'' = 4$. For this case consider the circular magnetic mode corresponding to $\zeta_1 a = 3.905 + 0.344i$. If ϵ' is constant ($= 4$) while ϵ'' tends to infinity, this mode approaches the TM_{02} mode in a perfectly conducting guide; but if ϵ' and ϵ'' tend to infinity while remaining equal to each other, the same mode approaches TM_{01} in a perfectly conducting guide. Presumably the TM_{01}-limit mode in the former case coincides with the TM_{02}-limit mode in the latter case; but the value of $\zeta_1 a$ for this mode is outside the range of our calculations at $\epsilon' = \epsilon'' = 4$. A similar interchange occurs between the TM_{11}-limit and TM_{12}-limit modes in the large guide, depending on whether ϵ' is constant or ϵ' tends to infinity with ϵ''. There is no evidence of any such phenomenon in the smaller guide of Tables I(d) and I(e); but the fact that it can occur means that the limit-mode designations of modes in a lossy waveguide are not entirely unambiguous. The phenomenon is not due to the presence of the helix, since a helix of zero pitch has no effect on circular magnetic modes.

Finally it is of interest to compare the propagation constants given by the approximate formula with those obtained by numerical solution of the characteristic equation. A reasonably typical case is provided by the TM_{02}-limit mode in a 2-inch guide at $\lambda_0 = 5.4$ mm with $\epsilon' = 4$, as in Table I(a). Exact and approximate results for βa vs αa and αa vs ϵ'' are plotted in Fig. 4. As the conductivity decreases, the attenuation con-

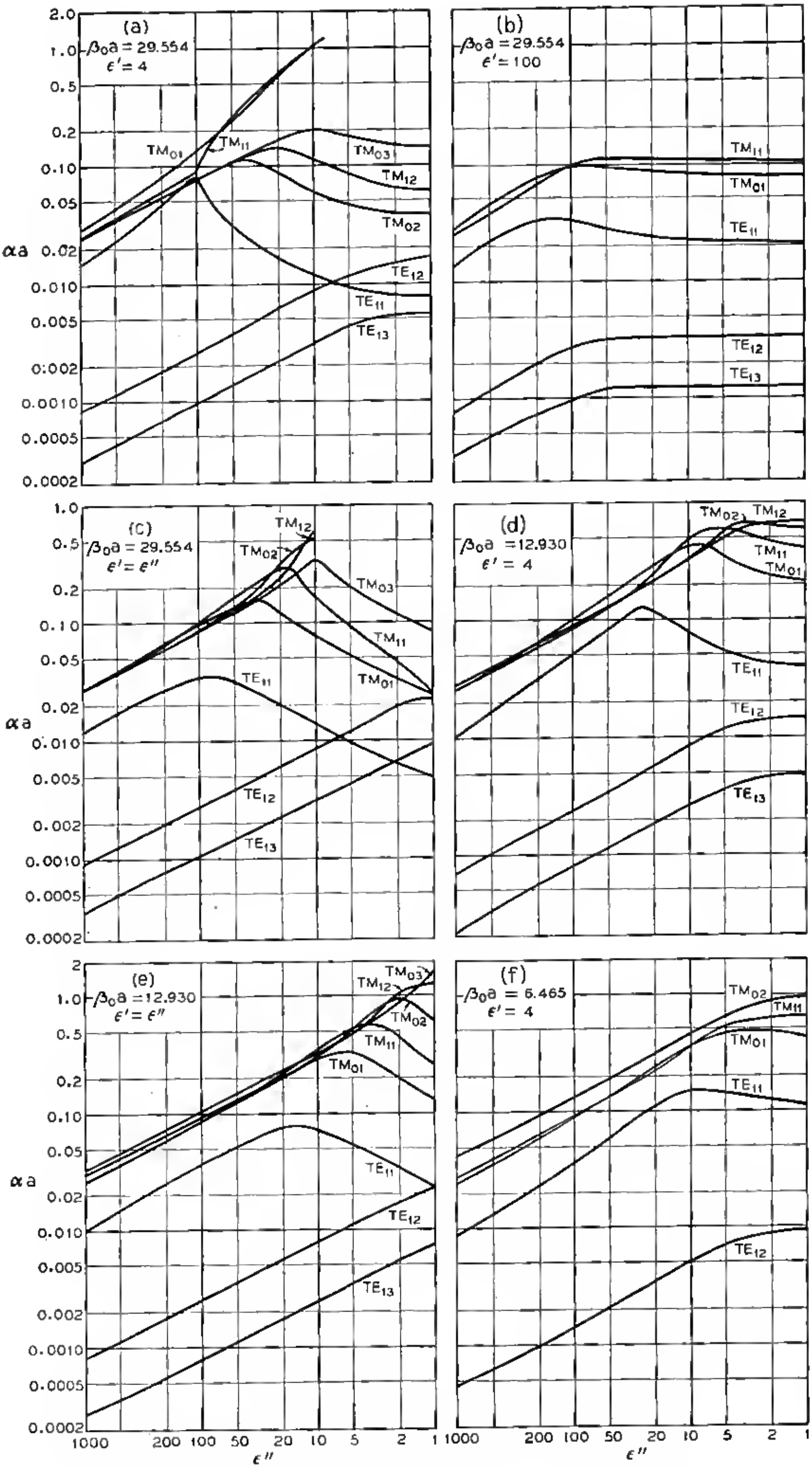


Fig. 3 — Attenuation constant as a function of jacket conductivity for modes in various helix waveguides.

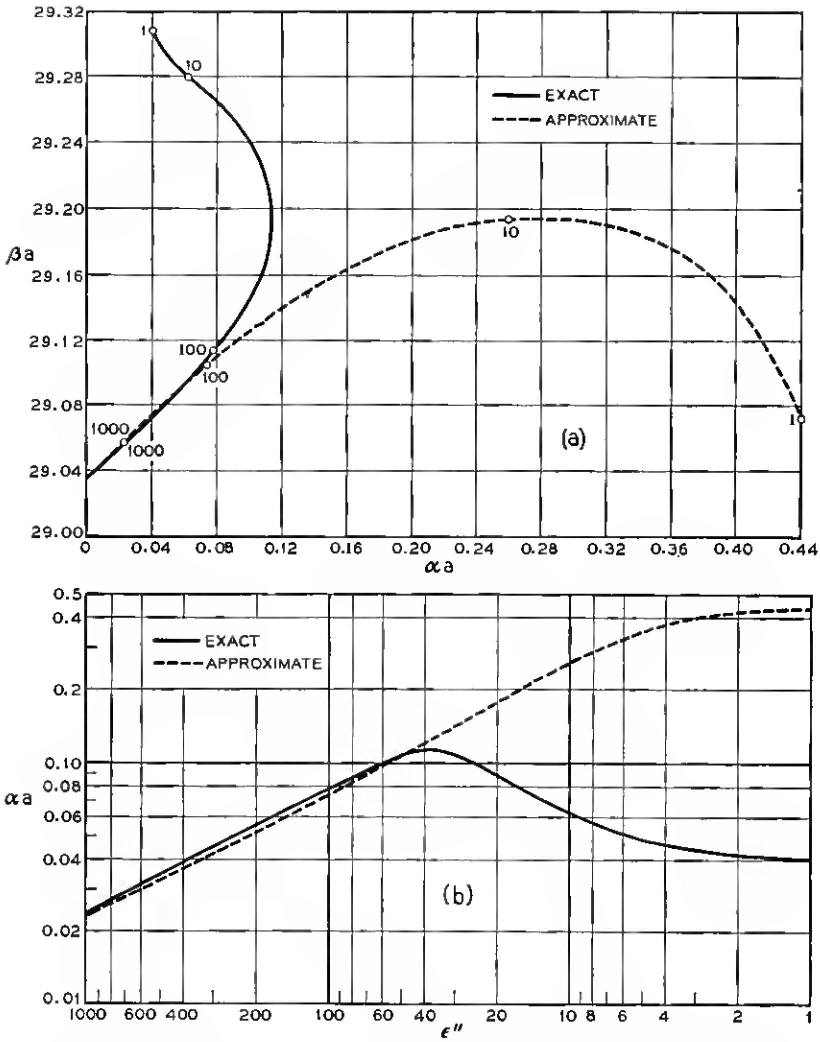


Fig. 4 — Comparison of exact and approximate formulas for the propagation constant of a typical mode (TM_{02}- limit in a guide with $\beta_0 a = 29.554$ and $\epsilon' = 4$).

stant first becomes larger, in all cases, than predicted by the approximate formula. For still lower conductivities the attenuation constant may pass through a maximum, as in the present example, and decrease again. The existence of a maximum in the attenuation vs conductivity curve is not indicated by the approximate formula.

V. DISCUSSION OF RESULTS

The dimensionless results of Section IV may easily be scaled to any desired operating wavelength, and the attenuation constants and guide wavelengths expressed in conventional units. If λ_0 is the free-space wavelength in centimeters, then the guide diameter d in inches, the attenuation constant α in db/meter, and the guide wavelength λ_g in centimeters are given by the following formulas:

$$d_{\text{in}} = 0.12532\,(\beta_0 a)(\lambda_0)_{\text{cm}}$$

$$\alpha_{\text{db/m}} = \frac{5457.5\,(\alpha a)}{(\beta_0 a)(\lambda_0)_{\text{cm}}}$$

$$(\lambda_g)_{\text{cm}} = \frac{(\beta_0 a)(\lambda_0)_{\text{cm}}}{(\beta a)}$$

Table II lists the guide diameters and the conversion factors for α and λ_g for the three values of $\beta_0 a$ used in Section IV, at frequencies corresponding to free-space wavelengths of 3.33 and 0.54 cm. The table also lists the number of propagating modes in a perfectly conducting guide as a function of $\beta_0 a$ (different polarizations are not counted separately).

When helix waveguide is used to reduce mode conversions, an important parameter is the ratio of the attenuation constant of any given unwanted mode to the attenuation constant of the TE_{01} mode. The theoretical attenuation constants of the TE_{01} mode at $\lambda_0 = 5.4$ mm in copper guides of various sizes are listed below:

Diameter	αa	αdb/m
2″	2.77×10^{-6}	9.47×10^{-4}
$\frac{7}{8}$″	1.50×10^{-5}	1.17×10^{-2}
$\frac{7}{16}$″	7.11×10^{-5}	1.11×10^{-1}

Table II — Conversion Factors for Attenuation Constants and Guide Wavelengths in Various Waveguides

$\beta_0 a$	Propagating modes	$\lambda_0 = 3.33$ cm			$\lambda_0 = 0.54$ cm		
		Diameter (inches)	α db/meter	λ_g cm	Diameter (inches)	α db/meter	λ_g cm
29.554	227	12.33	55.5 αa	$98.41/\beta a$	2.000	342 αa	$15.959/\beta a$
12.930	44	5.40	127 αa	$43.06/\beta a$	0.875	782 αa	$6.982/\beta a$
6.465	12	2.70	253 αa	$21.53/\beta a$	0.4375	1563 αa	$3.491/\beta a$

Referring to the values of αa listed in Table I, we see that the unwanted mode attenuations can be made to exceed the TE_{01} attenuation by factors of from several hundred to several hundred thousand in the large helix guide. The attenuation ratios are somewhat smaller in the smaller guide sizes.

The attenuation versus conductivity plots of Fig. 3 show that for many of the modes there is a value of jacket conductivity, depending on the mode, the value of $\beta_0 a$, and the jacket permittivity, which maximizes the attenuation constant. Since one is accustomed to think of the attenuation constant of a waveguide as an increasing function of frequency for all sufficiently high frequencies (except for circular electric waves), or as an increasing function of wall resistance, it is worth while to see why one should really expect the attenuation constant to pass through a maximum as the frequency is increased indefinitely in an ordinary metallic guide, or as the wall resistance is increased at a fixed frequency. The argument runs as follows:

Guided waves inside a cylindrical pipe may be expressed as bundles of plane waves repeatedly reflected from the cylindrical boundary.[11] The angle which the wave normals make with the guide axis decreases as the frequency increases farther above cutoff; and the complementary angle, which is the angle of incidence of the waves upon the boundary, approaches 90°. If the walls are imperfectly conducting, the guided wave is attenuated because the reflection coefficient of the component waves at the boundary is less than unity. The theory of reflection at an imperfectly conducting surface shows that the reflection coefficient of a plane wave polarized with its electric vector in the plane of incidence first decreases with increasing angle of incidence, then passes through a deep minimum, and finally increases to unity at strictly grazing incidence.[12] For a metallic reflector, the angle of incidence corresponding to minimum reflection is very near 90°. Inasmuch as all modes in circular guide except for the circular electric family have a component of $\vec{E}$ in the plane of incidence (the plane θ = constant), one would expect the attenuation constant of each mode to pass through a maximum at a sufficiently high frequency. For example, the TM_{01} mode in a 2-inch copper guide should have maximum attenuation at a free-space wavelength in the neighborhood of 0.1 mm (100 microns), assuming the dc value for the conductivity of copper. To find the actual maximum, of course, would require the solution of a transcendental equation as in Section IV.

The circular electric waves all have $\vec{E}$ normal to the plane of incidence.

[11] Reference 9, pp. 411–412.
[12] Reference 7, pp. 507–509.

For this polarization the reflection coefficient increases steadily from its value at normal incidence to unity at grazing incidence. Thus one bas an optical interpretation of the anomalous attenuation-frequency behavior of circular electric waves.

If instead of varying the frequency one imagines tbe wall resistance varied at a fixed frequency, he can easily convince himself that there usually exists a finite value of resistance which maximizes the attenuation constant of a given mode. An idealized illustrative example has been worked out by Schelkunoff.[13] He considers the propagatiou of transverse magnetic waves between parallel resistance sheets, and shows that if the sheets are far enough apart the attenuation constant increases from zero to a maximum and then falls again to zero, as the wall resistance is made to increase from zero to infinity. It may be instructive to consider that maximum power is dissipated in the lossy walls when their impedance is matched as well as possible to the wave impedance, looking normal to the walls, of the fields inside the guide.

In conclusion we mention a couple of theoretical questions which are suggested by the numerical results of Section IV.

(1) Limit modes. It has been seen that the limit which a given lossy mode approaches as the jacket conductivity becomes infinite may not be unique. Can rules be given for determining limit modes when the manner in which $| \epsilon' - i\epsilon'' |$ approaches infinity is specified?

(2) Behavior of modes as $\sigma \rightarrow 0$. It is known[14] that the number of true guided waves (i.e., exponentially propagating waves whose fields vanish at large radial distances from the guide axis) possible in a cylindrical waveguide is finite if the conductivity of the exterior medium is finite. The number is enormously large if the exterior medium is a metal; but the modes presumably disappear one by one as the conductivity is decreased. If the conductivity of the exterior medium is low enough and if its permittivity is not less than the permittivity of the interior medium, no true guided waves can exist. At what values of conductivity do the first few modes appear in a guide of given size, and how do their propagation constants behave at very low conductivities?

The complete theory of lossy-wall waveguide would appear to present quite a challenge to the applied mathematician. Fortunately the engineering usefulness of helix waveguide does not depend upou getting immediate answers to such difficult analytical questions.

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[13] Reference 9, pp. 484-489.

[14] G. M. Roe, The Theory of Acoustic and Electromagnetic Wave Guides and Cavity Resonators, Ph.D. thesis, U. of Minn., 1947, Section 2.

APPENDIX

APPROXIMATE SOLUTION OF THE CHARACTERISTIC EQUATION

The characteristic equation (6) of the helix guide may be written in the dimensionless form

$$\begin{aligned}\left(\zeta_1 a \tan\psi - \frac{nha}{\zeta_1 a}\right)^2 \frac{J_n(\zeta_1 a)}{J_n'(\zeta_1 a)} - (\beta_0 a)^2 \frac{J_n'(\zeta_1 a)}{J_n(\zeta_1 a)} \\ = \frac{\zeta_1 a}{\zeta_2 a}\left[\left(\zeta_2 a \tan\psi - \frac{nha}{\zeta_2 a}\right)^2 \frac{H_n^{(2)}(\zeta_2 a)}{H_n^{(2)\prime}(\zeta_2 a)} - (\beta_0 a)^2(\epsilon' - i\epsilon'')\frac{H_n^{(2)\prime}(\zeta_2 a)}{H_n^{(2)}(\zeta_2 a)}\right]\end{aligned} \tag{A1}$$

If $| \epsilon' - \epsilon'' |$ is sufficiently large, the right side of the equation is large and either $J_n(\zeta_1 a)$ or $J_n'(\zeta_1 a)$ is near zero. Let p denote a particular root of J_n or J_n'; then to zero order,

$$\begin{aligned} \zeta_1 a &= p \\ ha &= \beta_{nm} a = \beta_0 a(1 - \nu^2)^{1/2} \\ \zeta_2 a &= \beta_0 a(\epsilon' - i\epsilon'' - 1 - \nu^2)^{1/2} \end{aligned} \tag{A2}$$

where

$$\nu = p/\beta_0 a$$

Henceforth assume that

$$| \zeta_2 a | \gg | (4n^2 - 1)/8 | \tag{A3a}$$

and

$$| \zeta_2 a | \gg | n | \tag{A3b}$$

It is convenient to postulate both inequalities, even though the first is more restrictive than the second unless $| n | = 1$ or $| n | = 2$.

If (A3a) is satisfied, the Hankel functions may be replaced by the first terms of their asymptotic expressions, and

$$\frac{H_n^{(2)\prime}(\zeta_2 a)}{H_n^{(2)}(\zeta_2 a)} = -i$$

Eq. (A1) becomes

$$\left(\zeta_1 a \tan\psi - \frac{nha}{\zeta_1 a}\right)^2 \frac{J_n(\zeta_1 a)}{J_n'(\zeta_1 a)} - (\beta_0 a)^2 \frac{J_n'(\zeta_1 a)}{J_n(\zeta_1 a)} = \frac{i\zeta_1 a}{\zeta_2 a}\left[\left(\zeta_2 a \tan\psi - \frac{nha}{\zeta_2 a}\right)^2 + (\beta_0 a)^2(\epsilon' - i\epsilon'')\right]$$

It follows from (A3b), using the zero-order approximations (A2), that

$$| nha/\zeta_2 a | \ll | \beta_0 a(\epsilon' - i\epsilon'')^{1/2} |$$

so the characteristic equation finally takes the approximate form

$$\left(\zeta_1 a \tan \psi - \frac{nha}{\zeta_1 a}\right)^2 \frac{J_n(\zeta_1 a)}{J_n'(\zeta_1 a)} - (\beta_0 a)^2 \frac{J_n'(\zeta_1 a)}{J_n(\zeta_1 a)} = \frac{i\zeta_1 a}{\zeta_2 a}[(\zeta_2 a \tan \psi)^2 + (\beta_0 a)^2(\epsilon' - i\epsilon'')] \tag{A4}$$

Now let

$$\zeta_1 a = p + x, \qquad | x | \ll 1$$

where x is a small complex number. The normalized propagation constant becomes, to first order,

$$\begin{aligned} iha &= [(\zeta_1 a)^2 - (\beta_0 a)^2]^{1/2} \\ &= i\beta_0 a(1 - \nu^2)^{1/2} - i\nu x(1 - \nu^2)^{-1/2} \\ &= \alpha a + i(\beta_{nm} a + \Delta\beta a) \end{aligned}$$

where β_{nm} is the phase constant of the mode in a perfectly conducting guide, and the perturbation terms are

$$\alpha a + i\Delta\beta a = -\frac{i\nu x}{(1 - \nu^2)^{1/2}} \tag{A5}$$

For the TM_{nm} mode, let p be the m^{th} root of J_n ; then from Taylor's series, to first order in x,

$$J_n(\zeta_1 a) = J_n(p + x) = xJ_n'(p) \tag{A6}$$

Substituting (A6) into (A4), neglecting the first term on the left side of (A4), and replacing everything on the right side by its zero approximation according to (A2), one obtains

$$-\frac{(\beta_0 a)^2}{x} = \frac{ip\beta_0 a[(\epsilon' - i\epsilon'' - 1 + \nu^2) \tan^2 \psi + (\epsilon' - i\epsilon'')]}{(\epsilon' - i\epsilon'' - 1 + \nu^2)^{1/2}}$$

or

$$x = \frac{i(\xi + i\eta)}{\nu\left[1 + \left\{1 - \dfrac{1 - \nu^2}{\epsilon' - i\epsilon''}\right\} \tan^2 \psi\right]} \tag{A7}$$

where

$$\xi + i\eta = \frac{\left[1 - \dfrac{1 - \nu^2}{\epsilon' - i\epsilon''}\right]^{1/2}}{(\epsilon' - i\epsilon'')^{1/2}} \tag{A8}$$

It follows from (A5) and (A7) that for TM modes,

$$\alpha + i\Delta\beta = \frac{\xi + i\eta}{a(1 - \nu^2)^{1/2}\left[1 + \left\{1 - \dfrac{1 - \nu^2}{\epsilon' - i\epsilon''}\right\}\tan^2\psi\right]} \tag{A9}$$

where $\xi + i\eta$ is given by (A8).

For the TE_{nm} mode, let p be the m^{th} root of J_n'; then

$$J_n'(\zeta_1 a) = J_n'(p + x) = \frac{(n^2 - p^2)x}{p^2} J_n(p)$$

Equation (A4) yields

$$x = \frac{ip^2\nu}{(p^2 - n^2)} \frac{\left[\tan\psi - \dfrac{n(1 - \nu^2)^{1/2}}{p\nu}\right]^2 (\xi + i\eta)}{\left[1 + \left\{1 - \dfrac{1 - \nu^2}{\epsilon' - i\epsilon''}\right\}\tan^2\psi\right]}$$

and, using (A5), we have for TE modes,

$$\alpha + i\Delta\beta$$

$$= \frac{p^2}{(p^2 - n^2)} \frac{\nu^2}{a(1 - \nu^2)^{1/2}} \frac{\left[\tan\psi - \dfrac{n(1 - \nu^2)^{1/2}}{p\nu}\right]^2 (\xi + i\eta)}{\left[1 + \left\{1 - \dfrac{1 - \nu^2}{\epsilon' - i\epsilon''}\right\}\tan^2\psi\right]} \tag{A10}$$

where $\xi + i\eta$ is given by (A8).

In view of (A5), the condition that $|x| \ll 1$ is equivalent to

$$\frac{(1 - \nu^2)^{1/2}}{\nu} |\alpha a + i\Delta\beta a| \ll 1 \tag{A11}$$

In all the numerical cases treated in the present paper, the approximate formulas agree well with the exact ones provided that the left side of (A11) is not greater than about 0.1.

A condition which is usually satisfied in practice, although not strictly a consequence of the assumptions (A3) or (A11), is

$$\left|\frac{1 - \nu^2}{\epsilon' - i\epsilon''}\right| \ll 1$$

This final approximation leads to the simple equations (7a) and (7b) of Section III, namely:

TM_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{\xi + i\eta}{a(1 - \nu^2)^{1/2}[1 + \tan^2\psi]}$$

TE_{nm} *modes*

$$\alpha + i\Delta\beta = \frac{(\xi + i\eta)}{a(1 - \nu^2)^{1/2}} \frac{\nu^2 p^2}{(p^2 - n^2)} \frac{[\tan\psi - n(1 - \nu^2)^{1/2}/p\nu]^2}{[1 + \tan^2\psi]}$$

where

$$\xi + i\eta = (\epsilon' - i\epsilon'')^{-1/2}$$